

Differential Equation (First order First degree)

Homogeneous Equations:- A differential equation

of the form  $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

where  $f(x,y)$  and  $\phi(x,y)$  are homogeneous functions of  $x, y$  and of the same degree is said to be homogeneous.

Solving Rule :- Every homogeneous equation of the above type can be solved by putting  $y = vx$  so that  $\frac{dy}{dx} = v + x \frac{dv}{dx}$ .

After making the substitution for  $y$  and  $\frac{dy}{dx}$ , the given differential equation can be reduced to the form in which the variables are separable. Then after integration we replace  $y$  by  $\frac{y}{x}$  and the required result follows.

Ex. ① Solve  $x \frac{dy}{dx} - y = x \sqrt{x^2 + y^2}$

Soln. From the given eqn. we have

$$\frac{dy}{dx} = \frac{y + x \sqrt{x^2 + y^2}}{x}, \text{ Put } y = vx, \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2 x^2}}{x} = v + x \sqrt{1+v^2}$$

$$\Rightarrow x \frac{dv}{dx} = x \sqrt{1+v^2} \Rightarrow dx = \frac{dv}{\sqrt{1+v^2}} \Rightarrow \int dx = \int \frac{dv}{\sqrt{1+v^2}}$$

$$\Rightarrow x + c = \sinh^{-1} v \Rightarrow v = \sinh(x+c)$$

$$\Rightarrow \frac{y}{x} = \sinh(x+c) \Rightarrow y = x \sinh(x+c)$$

which is reqd. soln.

Ex 2 Solve  $(x^2 - y^2) \frac{dy}{dx} = 2xy$

Soln. Given  $(x^2 - y^2) \frac{dy}{dx} = 2xy$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \quad \text{--- (1)}$$

Put  $y = vx$ , so that  $\frac{dy}{dx} = v + x \frac{dv}{dx}$

Hence the eqn. (1) becomes,

$$v + x \frac{dv}{dx} = \frac{2x \cdot vx}{x^2 - v^2 x^2} = \frac{2v \cancel{x^2}}{\cancel{x^2}(1 - v^2)} = \frac{2v}{1 - v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v}{1 - v^2} - v = \frac{2v - v + v^3}{1 - v^2} = \frac{v + v^3}{1 - v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{1 - v^2}{v + v^3} dv = \frac{1 - v^2}{v(1 + v^2)} dv$$

Now, integrating, we get

$$\int \frac{dx}{x} = \int \frac{1 - v^2}{v(1 + v^2)} dv = \int \left( \frac{1}{v} - \frac{2v}{1 + v^2} \right) dv$$

$$\Rightarrow \log x = \log v - \log(1 + v^2) + K$$

$$= \log \frac{y}{x} - \log \left( 1 + \frac{y^2}{x^2} \right) + K$$

$$= (\log y - \log x) - \{ \log(x^2 + y^2) - \log x^2 \} + K$$

$$= \log y - \log x - \log(x^2 + y^2) + 2 \log x + K$$

$$= \log y + \log x - \log(x^2 + y^2) + K$$

$$\Rightarrow \log(x^2 + y^2) = \log y + \log x - \log x + K$$

$$\Rightarrow \log(x^2 + y^2) = \log y + K = \log y + \log c \quad [K = \log c]$$

$$\Rightarrow \log(x^2 + y^2) = \log cy$$

$$\Rightarrow x^2 + y^2 = cy$$

which is the required soln.